JOURNAL OF COMBINATORIAL THEORY, Series A 26, 249–268 (1979)

Collineations of Projective Planes of Order 10, Part I

S. H. WHITESIDES*

Tufts University, Medford, Massachusetts 02155

Communicated by the Managing Editors

Received April 28, 1977

No projective plane of order 10 has a collineation group of order 9 which fixes a 12-arc, a set of twelve points no three collinear, as a set. This fact, proved in Part I, is used to prove in Part II that the full collineation group of any projective plane of order 10 has order 1, 3, or 5.

I. INTRODUCTION

The purpose of this paper and its sequel is to prove that if a finite projective plane of order 10 exists, its collineation group has order 1, 3, or 5.

As Hughes [13] pointed out, it is easy to check that the results of [13] and Hughes [12] imply that 3, 5, and 11 are the only possible *prime* divisors of a collineation group of a plane of order 10. In [16], we have shown that, in fact, 11 cannot be a divisor. (The method developed there applies more generally to collineations of prime order $n + 1$ in planes of even order n.)

Whether or not planes of order 10 do exist is a question of considerable interest, as 10 is the smallest integer which is neither a prime power nor an order ruled out by the Bruck-Ryser Theorem [6]. Two general references on planes are Dembowski [9] and Hughes and Piper [14].

In Part I, we prove the following theorem.

THEOREM 1.1. *No projective plane of order* 10 *has a collineation group of order* 9 *which fixes a* 12-*arc as a set.*

This result was contained in the author's Ph. D. thesis [17]; the valuable advice of Professor R. H. Bruck is gratefully acknowledged.

In Part II, we use the result of Part I that groups of order 9 cannot fix 12-arcs to show that no plane of order 10 can have a collineation group of order 9. Upon noting that groups of order 15 and 25 cannot occur, we conclude that the full collineation group of any plane of order 10 has order 1, 3, or 5.

* Present address: Dartmouth College, Hanover, New Hampshire 03755.

0097-3165/79/030249–20$2.00/0

2. Possibilities for Groups of Order 9 Fixing a 12-Arc

A *k-arc* of a plane is a set of k points, no three collinear. The number of 6-arcs in any plane can be counted easily. Denniston [10] showed that all 6-arcs in a plane of order 10 are embeddable in 7-arcs. Furthermore, Bruck [5] has found that 8-arcs exist in any plane of order 10. It is not known whether 12-arcs must also exist in a plane of order 10. These 12-arcs precisely correspond to the vectors of weight 12 in the vector space over $GF(2)$ spanned by the rows of the 111×111 incidence matrix. (This vector space has been studied by Assmus [1], Assmus and Mattson [2], Assmus, Mattson, and Guza [3], Carter [8], Guza [11], and MacWilliams, Sloane and Thompson [15]. Bruen and Fisher [7] show that the main results of [10] and [15] are equivalent.) Bruck [4] suggested that the investigation of 12-arcs in planes of order 10 might be fruitful for the following reason. If a 12-arc $\mathcal{O}\!\mathcal{l}$ exists, every line containing a point of $\mathcal{O}\!\mathcal{l}$ must be a *secant* to $\mathcal{O}\!\mathcal{l}$, a line containing two of its points. Hence, every point not in $\mathcal{O}\!\mathcal{l}$ may be represented as the common intersection point of some six secants to $\mathcal{O}\!\mathcal{l}$. Consequently, if two collineations have the same action on a 12-arc which each one fixes as a set, then the two are identical.

Remark 2.1. As Hughes [13] pointed out, it is easy to show, using the results of Hughes [12], that a collineation of order 3 in a plane of order 10 has just two possible structures for its fixed points and lines:

(i) three non-collinear points, together with the three lines they determine, or

(ii) eight collinear points plus a point off the line through the eight, together with the lines determined by these nine points.

Remark 2.2. It follows from Remark 2.1 that if ϕ is a collineation of order 3 of a plane of order 10, and if ϕ fixes a 12-arc as a set, then ϕ fixes either no points or three points of the 12-arc.

From now on, we assume that Π is a projective plane of order 10 with a collineation group Γ of order 9 which fixes a 12-arc $\mathcal{O}\!\mathcal{l} = \{0,\dots, 8, a, b, c\}$ as a set, and we identify each element of Γ with its permutation of the elements of $\mathcal{O}\!\mathcal{l}$. We now list the possibilities for Γ.

Theorem 2.3. *Γ is isomorphic to one of the following groups;*

(i) $\Gamma_1 = \langle(012345678)\,(a)\,(b)\,(c)\rangle$;

(ii) $\Gamma_2 = \langle(012345678)\,(abc)\rangle$;

(iii) $\Gamma_3 = \langle(012)\,(354)\,(678)\,(a)\,(b)\,(c),\ (abc)\,(012)\,(345)\,(6)\,(7)\,(8)\rangle$;

(iv) $\Gamma_4 = \langle (012)(345)(678)(a)(b)(c), (036)(147)(258)(a)(b)(c) \rangle$;

(v) $\Gamma_5 = \langle (abc)(012)(345)(678), (036)(147)(258)(a)(b)(c) \rangle$.

Proof. If Γ is cyclic, then either $\Gamma \simeq \Gamma_1$ or $\Gamma \simeq \Gamma_2$.

Suppose Γ is elementary abelian and contains no elements with cycle structure 3^4. Then by Remark 2.2, all non-identity elements of Γ have cycle structure $3^3 1^3$. Conjugation of a non-identity element ϕ by an element not a power of ϕ either fixes or cyclically permutes the three 3-cycles of ϕ. In the former case, $\Gamma \simeq \Gamma_3$, and in the latter case, $\Gamma \simeq \Gamma_4$.

Now suppose Γ is elementary abelian and has an element γ with cycle structure 3^4. Then conjugation of γ by an element not a power of γ permutes the cycles of γ and fixes just one cycle; otherwise, Γ would contain non-identity elements fixing more than three points of $\mathcal{Cl}$, contradicting Remark 2.2. Hence, $\Gamma \simeq \Gamma_5$.

From now on, we assume that the points of $\mathcal{Cl}$ have been labelled so that Γ is equal to one of the five groups of Theorem 2.3.

3. Spreads

We repeat here two definitions of Whitesides [16], worded in terms of our present situation.

Definition 3.1. A *spread* is a set of six secants to a 12-arc, no two intersecting at a point of the 12-arc.

The above definition is essentially that of Bruck [4] and is not the notion of spread which appears in higher dimensional geometries.

Definition 3.2. A set $\mathcal{S}$ of spreads is *consistent* if any two distinct images of elements of $\mathcal{S}$ under the natural action of Γ have at most one line in common. A spread s is consistent if $\{s\}$ is consistent.

Remark 3.3. The six secants to 12-arc $\mathcal{Cl}$ which pass through a point p not in $\mathcal{Cl}$ form a spread which we say is *determined* by point p. Such a spread must be consistent. Furthermore, the spreads determined by the points of any subset of $\Pi \backslash \mathcal{Cl}$ form a consistent spread set which we say is determined by the subset. There are many more spreads than could be determined by points, however.

We often identify the points of $\Pi \backslash \mathcal{Cl}$ with the spreads they determine. For example, we refer to the "spreads" on a line, meaning the spreads determined by the points of the line. We denote the line determined by two points v and w of Π by vw, and we denote the point of intersection of two lines L and L' by $L \cap L'$.

We produce a contradiction to our assumption that there is a plane Π of order 10 with a collineation group Γ of order 9 fixing a 12-arc $\mathcal{O}$ by ruling out each of the possibilities for Γ listed in Theorem 2.3. Our method for each case is to derive one of the following contradictions:

(i) There is no consistent spread set, necessarily closed under the action of Γ, which could be determined by the points of $\Pi \backslash \mathcal{O}$ on the secants ab, bc, and ac;

(ii) There is no 9×9 matrix $M = [m_{ij}]$ whose entries could be defined by

$$m_{ij} = k \Leftrightarrow c(ai \cap bj) = ck, \qquad \text{where} \quad i, j, k \in \mathcal{O},\ 0 \leqslant i, j, k \leqslant 8. \quad (1)$$

The proof in Part II that the full collineation group of any plane of order 10 has order 1, 3, or 5 only requires that cases (i), (iii), and (iv) of Theorem 2.3 be ruled out. However, we consider all cases here, as the techniques developed can be extended to the study of planes of higher order n containing $n + 2$-arcs.

Remark 3.4. Suppose we identify the points of $\Pi \backslash \mathcal{O}$ with the spreads they determine and identify the lines of Π with the sets of spreads determined by their points. Let g be an element of the symmetric group Σ_{12}. If we use g to relabel the points of $\mathcal{O}$, we induce a relabelling of the secants of $\mathcal{O}$ in all spreads belonging to Π. Relative to this relabelling, Π has a group of collineations Γ^g fixing $\mathcal{O}$ as a set. Thus if $g \in N(\Gamma)$, the normalizer of Γ in $\sum_{12}$, the assumption that the spreads in some set $\mathcal{S}$ are points of Π is equivalent to the assumption that spreads in the set $\mathcal{S}g$, the image of $\mathcal{S}$ under the natural action of g, are points of Π.

LEMMA 3.5. *If λ is a collineation of order 3 fixing 12-arc $\mathcal{O}$, and if L is a line which intersects $\mathcal{O}$ either in two fixed points of λ or in two points of the same λ-orbit, then L contains no spreads which are fixed by λ.*

Proof. Suppose L contains a spread s which is fixed by λ.

If L intersects $\mathcal{O}$ in two fixed points, there is a third fixed point p in $\mathcal{O}$ by Remark 2.2. Certainly p does not lie on L. Line sp must contain a second point p' of $\mathcal{O}$, and p' cannot be fixed. Since line sp is fixed, $p'\lambda \in sp$. Thus p, p', and $p'\lambda \in sp$. This is a contradiction, since p, p', and $p'\lambda$ are distinct points of $\mathcal{O}$.

If, on the other hand, L intersects $\mathcal{O}$ in two distinct points q and $q\lambda$ of the same λ-orbit, then $L \cap L\lambda = q\lambda = s$, a contradiction.

Remark 3.6. If a spread s determined by a point of Π contains distinct lines L and L', and if Γ contains an element γ such that $L\gamma$ and $L'\gamma \in s$, then $s\gamma = s$.

Definition 3.7. Let the points of 12-arc $\mathfrak{A}$ be partitioned into four 3-element subsets, or *triangles*. Then spreads s and t belong to the same equivalence class of *types* relative to this partition if there is a permutation σ of the triangles such that for any triangles Δ and Δ' of the partition, s

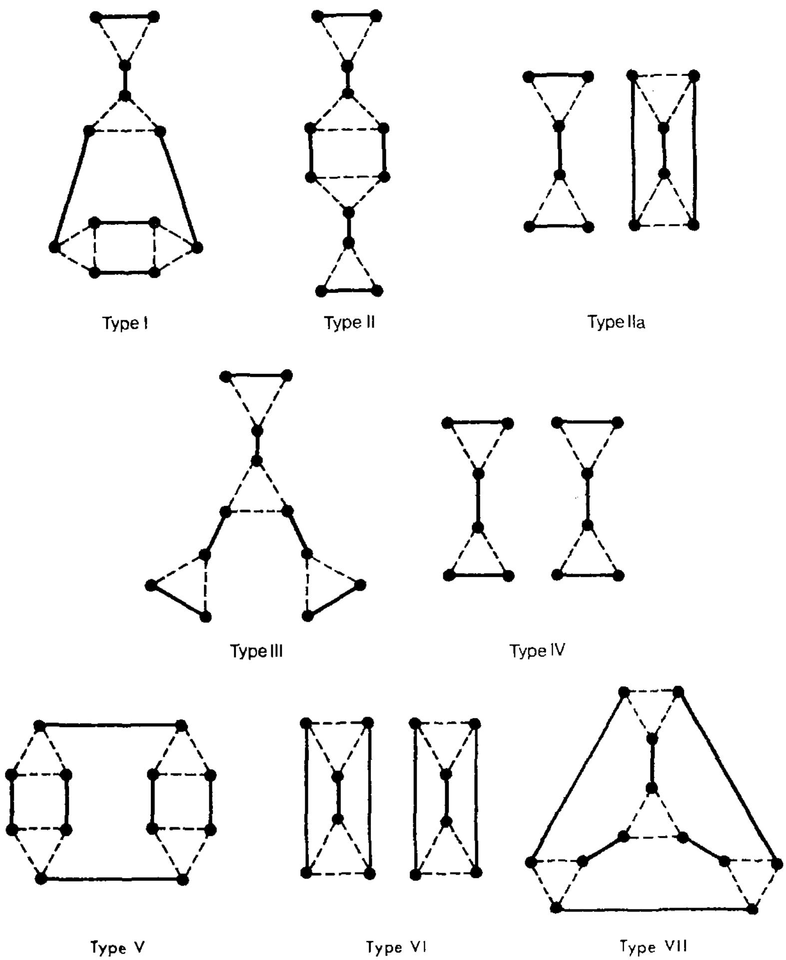

Fig. 1. Spread types. (In the Figures, solid lines represent spread elements and dashed lines represent triangle sides which are not spread elements.)

contains a line joining a point of Δ to a point of Δ' if and only if t contains a line joining a point of $\Delta\sigma$ to a point of $\Delta'\sigma$.

The following Lemma is very useful, as it enables us to do some counting which involves the numbers of spreads of various types relative to partitions of $\mathfrak{A}$ which arise from orbits under the action of Γ.

LEMMA 3.8. *Relative to a particular partition of 12-arc $\mathcal{O}\!\mathcal{l}$ into triangles, any spread belongs to one of the equivalence classes of types listed in Fig.* 1.

DEFINITION 3.9. If a line L contains a point of triangle Δ and a point of triangle Δ', then *L joins Δ to Δ'*.

DEFINITION 3.10. For any spread s of type I, II, or IIa, there is exactly one pair of triangles (Δ, Δ') which are joined by at least two lines of the spread. Spread s is *associated* with pair (Δ, Δ'), denoted $s \sim (\Delta, \Delta')$.

4. $\Gamma_1 = \langle(012345678)(a)(b)(c)\rangle$

We now assume that $\Gamma = \Gamma_1$ and reach a contradiction by showing that there is no spread set which could be determined by the points outside $\mathcal{O}\!\mathcal{l}$ on ab, bc, and ac.

Remark 4.1. Suppose a spread s on one of the fixed lines ab, bc, or ac contains two distinct lines ii' and jj', where $0 \leqslant i, i', j, j' \leqslant 8$. It follows from Lemma 3.5 that s is not fixed by any non-identity element of Γ. As a consequence of Remark 3.6, there is no k, $1 \leqslant k \leqslant 3$, such that $i + k = i' \bmod 9$ and $j + k = j' \bmod 9$.

Remark 4.2. It is easy to check that, as a consequence of Remark 4.1, each of the spreads $ab \cap co$, $ac \cap bo$, and $bc \cap ao$ has one of the following sets as a subset:

$$\{12, 57, 36, 48\}, \{23, 68, 47, 15\}, \{45, 81, 36, 72\},$$
$$\{78, 24, 36, 15\}, \{67, 13, 25, 48\}, \{12, 47, 58, 37\},$$
$$\{78, 35, 14, 26\}, \{34, 57, 82, 61\}, \{56, 24, 71, 83\}.$$

Remark 4.3. Suppose s and t are distinct spreads such that

$$i(i+k) \text{ and } (i+l)(i+m) \in s, \text{ and}$$
$$j(j+k) \text{ and } (j+l)(j+m) \in t, \qquad 0 \leqslant i, j, k, l, m \leqslant 8, \quad \text{addition mod } 9.$$

Then no consistent spread set contains both s and t, since there is an element of Γ which simultaneously sends $i(i+k)$ to $j(j+k)$ and $(i+l)(i+m)$ to $(j+l)(j+m)$.

We may now rule out $\Gamma = \Gamma_1$ by using Remark 4.3 to observe that no three of the nine sets of Remark 4.2 can belong to spreads of the same consistent spread set. For example, each of the first five sets contains a pair of lines of the form

$$i(i+1),\ (i+4)(i+6).$$

Hence no two of the first five sets can be subsets of spreads in the same consistent spread set.

5. $\Gamma_2 = \langle(012345678)(abc)\rangle$

We suppose that $\Gamma = \Gamma_2$ and reach a contradiction by showing that there can be no matrix $M = [m_{ij}]$ with entries defined by (1).

Entry m_{ij} has the property that lines ai, bj, and cm_{ij} are concurrent. Entry $m_{ii} = i$, and M is a Latin square. The entries are interrelated: if ai, bj, and ck are concurrent, then so are their images under the action of any element of Γ. Hence

$$m_{ij} = k \Leftrightarrow m_{k+1,i+1} = j+1,\ 0 \leqslant i, j, k \leqslant 8, \text{ addition mod } 9. \qquad (2)$$

As a consequence of (2), $m_{01} \neq 0, 1, 2, 5$, or 8.

A simple computation of moderate length shows that $m_{01} \neq 3$ or 4. Thus no spread determined by a point of Π contains either $\{a0, b1, c3\}$ or $\{a0, b1, c4\}$ as a subset. Since $(01)(28)(37)(46)(ab) \in N(\Gamma)$, it follows from Remark 3.4 that no spread determined by a point of Π contains either $\{a0, b1, c6\}$ or $\{a0, b1, c7\}$ as a subset. Hence $m_{01} \neq 6$ or 7, and M cannot exist.

6. $\Gamma_3 = \langle(012)(354)(678)(a)(b)(c), (abc)(012)(345)(6)(7)(8)\rangle$

In this section, we assume that $\Gamma = \Gamma_3$ and show that none of the possibilities for a consistent spread set determined by the points on ab, bc, and ac can be extended to a consistent set of spreads for Π.

Let 12-arc $\mathcal{O}$ be partitioned into triangles $\Delta_0 = \{0, 1, 2\}$, $\Delta_3 = \{3, 4, 5\}$, $\Delta_6 = \{6, 7, 8\}$, and $\Delta_a = \{a, b, c\}$. Throughout this section, "triangle" means one of these four sets.

Remark 6.1. Γ_3 has the following properties:

(i) Spreads in the same Γ_3-orbit have the same type, since the elements of Γ_3 fix the triangles as sets;

(ii) If h is a permutation of the points of one of the triangles and has cycle structure 3^1, and if k is a permutation of the points of a second triangle and also has cycle structure 3^1, then Γ_3 has a unique element containing cycles h and k;

(iii) The elements of Γ_3 which fix the points of any particular triangle form a subgroup of order 3.

LEMMA 6.2. *No points of Π determine spreads of types II, IIa, or VI.*

Proof. It follows from (ii) of Remark 6.1 that no spread of type II, IIa, or VI is consistent.

LEMMA 6.3. *Any line which is a side of a triangle contains three type IV spreads and six type I spreads.*

Proof. Let L be the side of a triangle, and consider the points at which L meets the nine lines which are sides of the other triangles. These points determine spreads which contain at least two lines which are sides of triangles and which therefore have type II, IIa, III, or IV. By Lemma 6.2, spreads of types II and IIa do not occur in Π.

Let L_{III} and L_{IV} be the number of spreads of type III and IV, respectively, on L. It follows from Lemma 3.5 and (iii) of Remark 6.1 that there is an element of Γ which fixes L and moves its spreads in orbits of length 3. Hence (i) of Remark 6.1 implies that

$$3 \mid L_{III} \qquad \text{and} \qquad 3 \mid L_{IV}. \tag{3}$$

Consider pairs (s, L'), where s is a spread on L, L' is a side of a triangle, and $L' \in s$. Counting in two ways, we find that

$$2L_{III} + 3L_{IV} = 9. \tag{4}$$

By (3) and (4), $L_{IV} = 3$ and $L_{III} = 0$. The remaining six spreads on L contain no other sides of triangles and so must be type I spreads.

LEMMA 6.4. *In addition to the* 12 *points of $\mathfrak{A}$, Π contains* 72 *type I spreads,* 9 *type IV spreads, and* 18 *type V spreads.*

Proof. Let the number of spreads in Π of type I, IV, and V be denoted by n_I, n_{IV}, and n_V, respectively.

Consider unordered triples (s, L, L'), where L and L' are distinct lines which join the same pair of triangles and belong to spread s of Π. Counting in two ways, we find that

$$\binom{4}{2} \cdot (9 \cdot 4)/2 = n_I + 2n_V. \tag{5}$$

By Lemma 6.3, the twelve lines which are sides of triangles each contain six type I spreads. Since each type I spread of Π lies on exactly one triangle side, $n_I = 72$. From (5), $n_V = 18$.

By counting in two ways the number of pairs (s, L), where s is a type IV spread of Π, L is a side of a triangle, and $L \in s$, we find, by Lemma 6.3, that $n_{IV} = 9$.

Remark 6.5. $N(\Gamma)$ contains (036) (245) (157). This element fixes the points of Δ_a while cyclically permuting Δ_0, Δ_3, and Δ_6. By Remark 3.4, we may assume that the nine type IV spreads of Π have the form shown in Fig. 2.

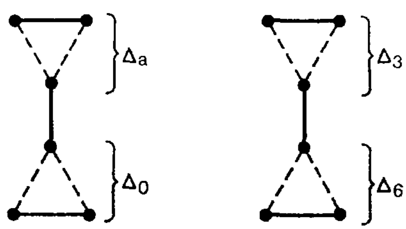

FIG. 2. Type IV spreads of Π for $\Gamma = \Gamma_3$.

LEMMA 6.6. *If the type IV spreads of Π are as in Fig. 2, then the type V spreads of Π are the only spreads of Π containing a pair of lines joining Δ_a to Δ_0. The type V spreads of Π have only the forms given in Fig.* 3.

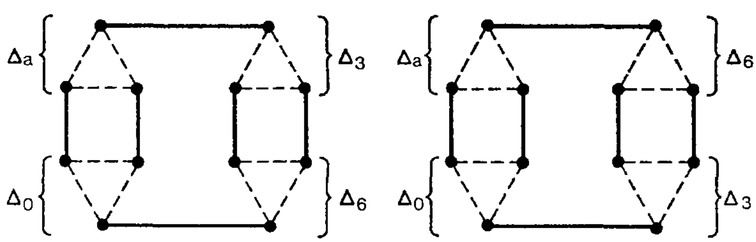

FIG. 3. Type V spreads of Π for $\Gamma = \Gamma_3$.

Proof. Consider unordered triples (s, L, L'), where L and L' are distinct lines which join Δ_a to Δ_0 and belong to spread s of Π. There must be $(9 \cdot 4)/2$ such triples, and certainly no spread in these triples has type IV.

We claim that no spread in these triples has type I. There are nine lines in Π which join Δ_3 to Δ_6. Each of the nine type IV spreads of Π contains such a line, and no such line can belong to two type IV spreads of Π. Each type IV spread of Π contains a side of Δ_3 and a side of Δ_6, so each line joining Δ_3

to Δ_6 meets a side of Δ_6 at a type IV spread. Therefore, no spread in a triple (s, L, L') has type I.

It now follows from Lemma 6.4 that the 18 triples (s, L, L') arise only from the 18 type V spreads of Π. Therefore, the type V spreads of Π contain two lines joining Δ_a to Δ_0, and they can have only the forms shown in Fig. 3.

LEMMA 6.7. *Nine type V spreads of Π contain a line joining Δ_a to Δ_3, and nine type V spreads of Π contain a line joining Δ_a to Δ_6.*

Proof. Consider triples (s, L, L'), where line L joins Δ_a to Δ_3, line L' joins Δ_0 to Δ_6, and L and L' belong to spread s of Π. There must be 81 such triples, none arising from a type IV spread.

Consider side ab of Δ_a. It follows from Lemma 6.3 that ab has three type IV spreads containing a line joining c to Δ_0, three type I spreads containing a line joining c to Δ_3, and three type I spreads containing a line joining c to Δ_6. The three spreads on ab which contain a line joining c to Δ_3 have two lines joining Δ_0 to Δ_6. Thus ab contributes six triples (s, L, L'). In fact, similar arguments show that each side of each triangle contributes six triples. Since a type I spread of Π lies on exactly one line which is a side of a triangle, 72 triples arise from type I spreads. Therefore, exactly nine type V spreads of Π contain a line joining Δ_a to Δ_3 and a line joining Δ_0 to Δ_6. By Lemma 6.6, the remaining nine type V spreads of Π must contain a line joining Δ_a to Δ_6 and a line joining Δ_0 to Δ_3.

Let $ab \cap c0$, $ab \cap c3$, and $ab \cap c6$ be denoted by s_0, s_3, and s_6, respectively. Since Γ is transitive on the lines joining Δ_a to Δ_0, s_0 has type IV by (i) of Remark 6.1. Since $(345) \in N(\Gamma)$ and $(678) \in N(\Gamma)$, we may assume that the line of s_0 which joins Δ_3 to Δ_6 is 56. Hence we may now assume that

$$s_0 = \{ab, c0, 12, 34, 56, 78\}. \tag{6}$$

A routine computation using (6) establishes the following Lemma.

LEMMA 6.8. *The possibilities for s_3, s_6 which meet the requirement that $\{s_0, s_3, s_6\}$ be a consistent set of spreads are*:

$$\{ab, c3, 04, 57, 28, 16\}, \{ab, c6, 18, 24, 05, 37\};$$
$$\{ab, c3, 57, 08, 26, 14\}, \{ab, c6, 04, 15, 28, 37\};$$
$$\{ab, c3, 24, 57, 18, 06\}, \{ab, c6, 14, 25, 08, 37\};$$
$$\{ab, c3, 05, 18, 27, 46\}, \{ab, c6, 03, 25, 17, 48\};$$
$$\{ab, c3, 15, 28, 07, 46\}, \{ab, c6, 05, 13, 27, 48\};$$
$$\{ab, c3, 25, 08, 17, 46\}, \{ab, c6, 15, 23, 07, 48\}.$$

The action of Γ on spreads s_0, s_3, and s_6 determines the remaining spreads on secants ab, bc, and ac. We now show that none of the possibilities listed in Lemma 6.8 can be extended to a consistent set of spreads for Π.

LEMMA 6.9. *If* $a0 \cap b1 = t$ *and* $a0 \cap b2 = t'$, *then the pair of spreads* t, t' *must be one of the following*:

$$\{a0, b1, c3, 28, 46, 57\}, \{a0, b2, c7, 14, 38, 56\};$$
$$\{a0, b1, c3, 28, 47, 56\}, \{a0, b2, c7, 14, 36, 58\};$$
$$\{a0, b1, c8, 23, 46, 57\}, \{a0, b2, c4, 17, 38, 56\};$$
$$\{a0, b1, c8, 23, 47, 56\}, \{a0, b2, c4, 17, 36, 58\}.$$

Proof. By Lemma 6.6, t and t' are type V spreads. Obviously, there are 36 type V spreads containing $a0$ and $b1$ and 36 type V spreads containing $a0$ and $b2$, where these spreads are not necessarily determined by points of Π. Since $\{c0, 56\} \subset s_0$ and s_0 is a type IV spread of Π, no image of this subset can be a subset of a type V spread of Π. This immediately eliminates 16 of the 36 type V spreads containing $a0$ and $b1$ and 16 of the 36 type V spreads containing $a0$ and $b2$ as possible spreads of Π.

Since no element of Γ takes the unordered pair $(a0, b1)$ to the unordered pair $(a0, b2)$, t and t' are not in the same Γ-orbit. Thus one of t and t' contains a line joining c to Δ_3, and the other contains a line joining c to Δ_6.

Spreads t and t' each contain four subsets $\{L, L'\}$, where L joins Δ_a to Δ_0 and L' joins Δ_3 to Δ_6. No such subset of t can be the image under the action of Γ of such a subset of t'.

It is now a routine computation to establish the Lemma.

We now use Lemmas 6.8 and 6.9 to observe that the set of spreads $\{s_3, s_6, t, t'\}$ cannot be consistent. Let $\theta = (012)\,(345)\,(678)\,(a)\,(b)\,(c)$. If $\{c3, 57\} \subset s_3$ and $\{c6, 37\} \subset s_6$, then

$$\{c3, 57\}, \{c4, 36\}, \{c7, 85\}, \{c8, 64\} \not\subset t \quad \text{or} \quad t', \tag{7}$$

as these three latter sets are subsets of $s_3\theta^2$, $s_6\theta$, and $s_6\theta^2$, respectively. Similarly, if $\{c3, 46\} \subset s_3$ and $\{c6, 48\} \subset s_6$, then

$$\{c3, 46\}, \{c4, 58\}, \{c7, 36\}, \{c8, 57\} \not\subset t \quad \text{or} \quad t'. \tag{8}$$

From (7) and (8), we see that $\{s_3, s_6, t, t'\}$ cannot be a consistent spread set.

7. $\Gamma_4 = \langle(012)(345)(678)(a)(b)(c), (036)(147)(258)(a)(b)(c)\rangle$

Here, we assume that $\Gamma = \Gamma_4 = \langle\lambda, \theta\rangle$, where

$\lambda = (036)\,(147)\,(258)\,(a)\,(b)\,(c)$ and $\theta = (012)\,(345)\,(678)\,(a)\,(b)\,(c)$.

We partition $\mathcal{A}$ into triangles

$$\Delta_a = \{a, b, c,), \Delta_0 = \{0, 3, 6\}, \Delta_1 = \{1, 4, 7\}, \quad \text{and} \quad \Delta_2 = \{2, 5, 8\},$$

and show that there is no consistent spread set for the spreads on the sides of Δ_a. The present triangles Δ_a, Δ_0, Δ_1, and Δ_2 do not form the same partition as that of Section 6. As before, however, the elements of Γ permute the triangles, so spreads in the same Γ-orbit have the same type.

Remark 7.1. It follows from Lemma 3.5 that the nine spreads on any side of Δ_a form a single Γ-orbit. Therefore, these spreads have the same type.

LEMMA 7.2. *The spreads on the sides of Δ_a have type II.*

Proof. Let L be a particular side of Δ_a. There are nine pairs (s, L') such that L' belongs to spread s on L, and L' is a side of Δ_0, Δ_1, or Δ_2. Therefore, it follows from Remark 7.1 that the spreads on L either all have type II or all have type IIa.

To show that the spreads on the sides of Δ_a do not have type IIa, we count unordered triples (s, M, M'), where distinct lines M and M' belong to spread s on a side of Δ_a, and M and M' join the same pair of triangles.

Since the spreads on a side of Δ_a form a single Γ-orbit, it follows from Remark 3.6 that M and M' cannot belong to the same Γ-orbit. Hence, given a line M which joins two triangles Δ_i and Δ_j, $0 \leqslant i \neq j \leqslant 2$, there are only two choices for an M' which joins Δ_i and Δ_j but does not belong to the Γ-orbit of M. However, M and M' may not intersect at a spread on a side of Δ_a. Therefore, there are at most $(3 \cdot 9 \cdot 2)/2 = 27$ unordered triples (s, M, M').

Now let n_{II} and n_{IIa} be the number of type II and type IIa spreads, respectively, on the sides of Δ_a. The number of unordered triples (s, M, M') is given by $n_{II} + 3n_{IIa}$, and $n_{II} + n_{IIa} = 27$. Since $9 \mid n_{II}$ and $9 \mid n_{IIa}$, $n_{IIa} = 0$. ∎

LEMMA 7.3. *Each side of Δ_a contains a spread of the form given in Fig.* 4.

Proof. Any side of Δ_a meets the line determined by 0 and the third point of Δ_a at some point which determines a spread s of Π. By Lemma 7.2, s has type II.

It follows from Remark 3.6 that for $0 \leqslant i \leqslant 8$, $\{3(\mathrm{i}\lambda), 6(\mathrm{i}\lambda^2)\} \not\subset s$. Hence $\{3(i\lambda^2), 6(i\lambda)\} \subset s$ for some $i \neq 0, 3, 6$ and $0 \leqslant i \leqslant 8$.

Remark 7.4. The line ij of Fig. 4 cannot lie in the Γ-orbit of either line $6(\mathrm{i}\lambda)$ or line $3(i\lambda^2)$, by Remark 3.6.

Remark 7.5. If $ij = 12$, then any spread of the form shown in Fig. 4 has type IIa relative to the partition Δ_a, $\{0, 1, 2\}$, $\{3, 4, 5\}$, and $\{6, 7, 8\}$. An argument analogous to the proof of Lemma 7.2 shows this is not the case.

By using Remarks 7.4 and 7.5, one may check easily that the only possibilities for the ordered pair (i, j) of points i and j of Fig. 4 are

$$(i, j) = (4, 8), (7, 5), (5, 7), \quad \text{or} \quad (8, 4). \tag{9}$$

Suppose u, v, and w are spreads of the form given in Fig. 4, and suppose $(i, j) = (4, 8)$, $(7, 5)$, and $(5, 7)$ for u, v, and w, respectively. Then

$$\{67, 31, 48, 25\} \subset u, \{61, 34, 75, 28\} \subset v, \quad \text{and} \quad \{68, 32, 57, 14\} \subset w. \tag{10}$$

From (10), note that $\{25, 67\} \subset u$, $v\lambda$, and $w\theta$. Hence no two of u, v, and w belong to Π. Since there are only four possibilities in (9), there is no consistent spread set for the spreads on the sides of Δ_a.

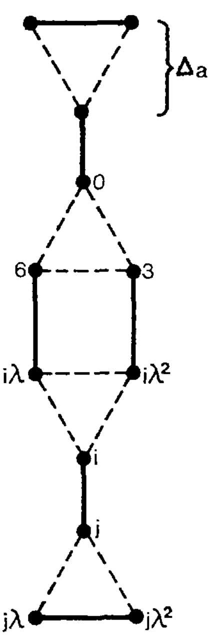

FIG. 4. Type II spread on a side of Δ_a, $\Gamma = \Gamma_4$.

8. $\Gamma_5 = \langle(abc)(012)(345)(678), (036)(147)(258)(a)(b)(c)\rangle$

In this final section, we assume that $\Gamma = \Gamma_5 = \langle\lambda, \theta\rangle$, where $\lambda = (036)(147)(258)(a)(b)(c)$ and $\theta = (abc)(012)(345)(678)$. Let 12-arc $\mathcal{O}$ be partitioned, as in Section 7, into triangles

$$\Delta_a = \{a, b, c\}, \Delta_0 = \{0, 3, 6\}, \Delta_1 = \{1, 4, 7\}, \quad \text{and} \quad \Delta_2 = \{2, 5, 8\},$$

so that spreads in the same Γ-orbit have the same type. We show that no consistent spread set for the spreads on the sides of Δ_a exists.

LEMMA 8.1. *All sides of Δ_a contain the same number of spreads of a given type, and each side contains three λ-orbits of length* 3, *namely*:

(i) *two orbits of type I spreads and one of type IV spreads, or*

(ii) *three orbits of type II spreads, or*

(iii) *one orbit of type I spreads, one of type II spreads, and one of type III spreads.*

Proof. Choose a side L of Δ_a, and let L_I, L_{II}, L_{IIa}, L_{III}, and L_{IV} be the number of spreads of type I, II, IIa, III, and IV, respectively, on L. Since θ cyclically permutes the sides of Δ_a, these numbers do not depend on the choice of L. Of course,

$$9 = L_I + L_{II} + L_{IIa} + L_{III} + L_{IV}. \tag{11}$$

By counting in two ways the pairs (s, L'), where L' belongs to spread s on L, and L' is a side of Δ_0, Δ_1, or Δ_2, we find that

$$9 = L_{II} + L_{IIa} + 2L_{III} + 3L_{IV}. \tag{12}$$

Consider unordered triples (t, M, M'), where M and M' are distinct lines belonging to a spread t on a side of Δ_a, and M and M' join the same pair of triangles. By Lemma 3.5 and Remark 3.6, M and M' cannot belong to the same λ-orbit, so counting triples (t, M, M') in two ways gives

$$27 \geqslant 3(L_I + L_{II} + 3L_{IIa}). \tag{13}$$

Since, by Lemma 3.5, λ fixes no spreads on the sides of Δ_a, 3 must divide all the parameters. This fact and (11), (12), and (13) yield cases (i), (ii), and (iii).

For $0 \leqslant i \leqslant 8$, let s_i be the spread determined by the point $ab \cap ci$. Spreads s_0, s_1, and s_2 lie in separate λ-orbits, so that all spreads on ab can be determined from these three. $N(\Gamma)$ contains λ and (012) (345) (678), which fixes a, b, and c and cyclically permutes sets Δ_0, Δ_1, and Δ_2. By Remark 3.4, we may assume that s_0 is a type IV spread in case (i) and a type I spread in case (iii). Of course, s_0 is a type II spread in case (ii).

We now rule out each of the three cases given in the Lemma. We handle case (i) by counting arguments. In cases (ii) and (iii), we show that the set of spreads $\{s_0, s_1, s_2\}$ cannot be consistent.

Case (i). *Each side of Δ_a contains two λ-orbits of type I spreads and one λ-orbit of type IV spreads.*

Spread s_0 by assumption is a type IV spread. Certainly s_0 belongs to a Γ-orbit of length 9.

Suppose L and L' are sides of different triangles. By counting triples (s, L, L'), where s is a spread of Π containing L and L', we see that $L \cap L'$ is a point in the orbit of s_0. It follows that Π contains nine type IV spreads and 72 type I spreads. All other spreads in Π have type V, VI, or VII.

Let n_I, n_{IV}, n_V, n_{VI}, and n_{VII} be the numbers of spreads in Π of types I, IV, V, VI, and VII, respectively. We have found that

$$n_I = 72, n_{IV} = 9, \qquad \text{and so} \qquad n_V + n_{VI} + n_{VII} = 18. \tag{14}$$

By counting unordered triples (s, L, L'), where s is a spread of Π containing distinct lines L and L' which join the same pair of triangles, we find that

$$n_I + 2n_V + 6n_{VI} = \tbinom{4}{2} \cdot 9 \cdot 4/2 = 108. \tag{15}$$

By counting unordered triples (s, L, L'), where s is a spread of Π containing lines L and L' which join the triangles in pairs, we find that

$$2n_I + n_{IV} + 5n_V + 9n_{VI} + 3n_{VII} = \tbinom{4}{2} \cdot 9 \cdot 9 = 486. \tag{16}$$

However, (14), (15), and (16) produce a contradiction.

Case (ii). *Each side of Δ_a contains three λ-orbits of type II spreads.*

Since λ fixes all four triangles, spreads in the same λ-orbit on ab are associated with the same pair of triangles.

Remark 8.2. Spreads in different λ-orbits on ab must be associated with different pairs of triangles for the following reason. For each pair (Δ_j, Δ_k), $0 \leqslant j \neq k \leqslant 2$, there are nine pairs (s_i, L), where $0 \leqslant i \leqslant 8$, $L \in s_i$, and L joins Δ_j to Δ_k.

Since $(ab)(36)(48)(57)(12) \in N(\Gamma)$, we may assume that $s_0 \sim (\Delta_0, \Delta_1)$. Then by Remark 8.2, $s_1 \sim (\Delta_1, \Delta_2)$ and $s_2 \sim (\Delta_2, \Delta_0)$. The forms of s_0, s_1, and s_2 are given in Fig. 5.

We may make some further assumptions about s_0. Since $(147)(285) \in N(\Gamma)$, we may assume that

$$31 \in s_0, \qquad \text{and so, by Remark 3.6,} \qquad 67 \in s_0. \tag{17}$$

From Fig. 5 and (17), we see that one of 42, 45, and 48 belongs to s_0 and that one belongs to s_1. Suppose, for the moment, that $48 \in s_0$, so $48 \notin s_1$. If

$45 \in s_1$, then $72 \in s_1$, contradicting $(48)\lambda = 72 \in s_0\lambda$. Therefore, $42 \in s_1$ and hence $78 \in s_1$. This contradicts a consequence of (17), that $\{42, 78\} \subset s_0\theta$. Therefore, $48 \notin s_0$, so either $42 \in s_0$ or $45 \in s_0$. Since $(36)\,(17)\,(25) \in N(\Gamma)$, we may assume that in fact, $42 \in s_0$. This assumption and (17) yield

$$s_0 = \{ab, c0, 31, 67, 42, 58\}. \tag{18}$$

We proceed to determine the lines in s_1.

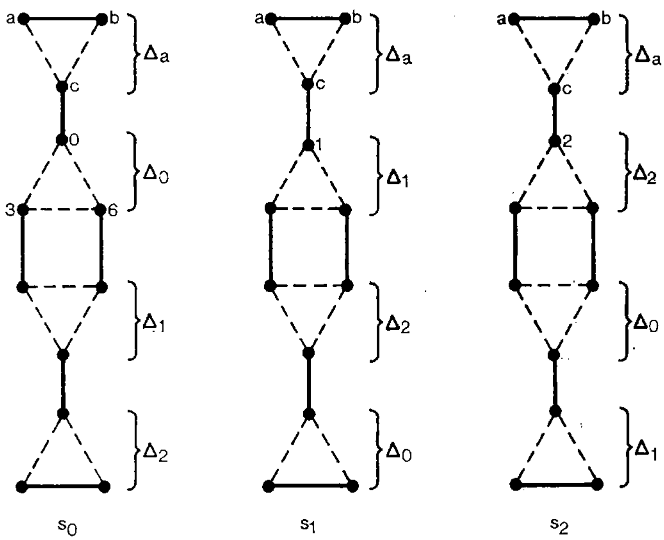

FIG. 5. Spreads s_0, s_1, and s_2 for $\Gamma = \Gamma_5$, case (ii).

We claim that of the possibilities 42, 45, and 48, line $45 \in s_1$. By (18), $42 \notin s_1$. Suppose $48 \in s_1$. Then $75 \in s_1$, so $(75)\,\lambda^2 = 42 \in s_1\lambda^2 = s_7$. This contradicts the fact that $42 \in s_0$. Therefore,

$$45 \in s_1, \qquad \text{and so} \qquad 72 \in s_1. \tag{19}$$

From (19) and Fig. 5, we see that either 80, 83, or 86 belongs to s_1. We claim that $86 \in s_1$. Consider the spread $s_2\theta$, which is associated with (Δ_0, Δ_1). By Remark 3.6, either $\{31, 67\}$, $\{34, 61\}$, or $\{37, 64\} \subset s_2\theta$. By (18), $\{31, 67\} \not\subset s_2\theta$. As a consequence of (19), $\{34, 61\} \subset s_1\theta^2$. Therefore, $\{37, 64\} \subset s_2\theta$. This means that $83 = (64)\,\theta^{-1} \in s_2$ and $80 = (37)\,\theta^2\lambda \in s_5$. Consequently, $86 \in s_1$. This fact and (19) give

$$s_1 = \{ab, c1, 45, 72, 86, 03\}. \tag{20}$$

Now consider the lines of s_2. In the process of deriving s_1, we have just shown that $\{37, 64\} \subset s_2\theta$. Therefore,

$$\{(37)\,\theta^{-1}, (64)\,\theta^{-1}\} = \{56, 83\} \subset s_2 . \tag{21}$$

From Fig. 5 and (21), we see that either 01, 04, or 07 $\in s_2$. By (18), (31) $\lambda^2 =$ 07 $\in s_6$ and (67)$\lambda =$ 01 $\in s_3$. Therefore, 04 $\in s_2$, and this fact together with (21) yields

$$s_2 = \{ab, c2, 83, 56, 04, 17\}. \tag{22}$$

From (22), $\{(83)\,\theta\lambda^2, (17)\,\theta\lambda^2\} = \{31, 58\} \subset s_2\theta\lambda^2$. This contradicts (18).

Case (iii). *Each side of Δ_a contains one λ-orbit of type I spreads, one of type II spreads, and one of type III spreads.*

Consider spread s_0, which, without loss of generality, we assume to have type I. It follows from Remark 3.6 that the lines of s_0 which join Δ_1 to Δ_2 are not in the same λ-orbit. Since (12) (45) (78) $(ab) \in N(\Gamma)$, we may assume that s_0 contains a line which joins point 6 to Δ_1. Hence s_0 has the form shown in Fig. 6. Since (147) (285) $\in N(\Gamma)$, we may assume that in fact, 61 $\in s_0$.

Fig. 6 shows that either 32, 35, or 38 $\in s_0$. We claim we may assume that

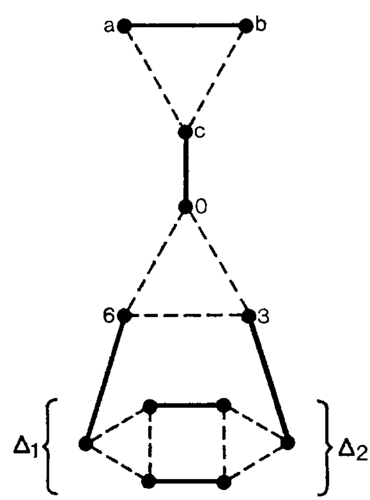

FIG. 6. Spread s_0 for $\Gamma = \Gamma_5$, case (iii).

35 $\in s_0$. Suppose, instead, that 32 $\in s_0$. Then 48 $\in s_0$ and $\{32, 48, 61\} \subset s_0$. Hence $\{(32)\,\lambda\theta^2, (48)\,\lambda\theta^2\} = \{48, 61\} \subset s_0\lambda\theta^2$, which contradicts $\{48, 61\} \subset s_0$. While either 35 or 38 may belong to s_0, these possibilities are equivalent, as (ab) (36) (187245) $\in N(\Gamma)$. We assume 35 $\in s_0$, so

$$s_0 = \{ab, c0, 61, 36, 42, 78\}. \tag{23}$$

In light of (23), the type II spreads on ab cannot be associated with (Δ_1, Δ_2), as this would contradict the fact that there are nine pairs (s_i, L), where $0 \leqslant i \leqslant 8$, $L \in s_i$, and L joins Δ_1 to Δ_2.

Suppose, for the moment, that s_1 has type II. Since $s_1 \sim (\Delta_1, \Delta_0)$, there is a $v \in \Delta_0$ such that $4v \in s_1$. Then it follows from Remark 3.6 that $7(v\lambda^2) \in s_1$. If $v = 3$, then $\{43, 70\} \subset s_1$. This contradicts a consequence of (23), that $\{(78)\,\lambda^2\theta^2, (42)\,\lambda^2\theta^2\} = \{43, 70\} \subset s_0\lambda^2\theta^2$. Hence $v \neq 3$. On the other hand, if v is either 0 or 6, then the λ-orbit of either $4v$ or $7(v\lambda^2)$, respectively, contains 61. This cannot occur, as $61 \in s_0$. Therefore, s_1 must have type III.

Now consider s_2, which must have type II. Since s_2 must be associated with (Δ_0, Δ_2), it has the form shown in Fig. 7, and there is a $w \in \Delta_0$ such

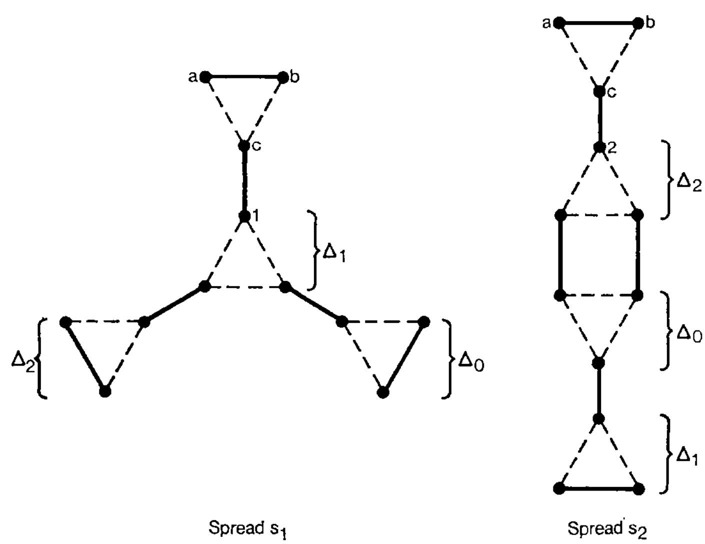

FIG. 7. Spreads s_1 and s_2 for $\Gamma = \Gamma_5$, case (iii).

that $5w \in s_2$. By Remark 3.6, $8(w\lambda^2) \in s_2$. In fact, $w = 6$, because neither $5w$ nor $8(w\lambda^2)$ may lie in the λ-orbit of $35 \in s_0$. Hence $\{56, 83\} \subset s_2$, and either 01, 04, or $07 \in s_2$. However, $04 \notin s_2$, because $04 = (61)\lambda \in s_0\lambda$. If $07 \in s_2$, then $\{56, 07\} \subset s_2$, which contradicts the consequence of (23), that $\{(61)\,\lambda^2\theta^2, (42)\,\lambda^2\theta^2\} = \{56, 07\} \subset s_0\lambda^2\theta^2$. Hence $01 \in s_2$, and

$$s_2 = \{ab, c2, 56, 83, 01, 47\}. \tag{24}$$

Finally, consider s_1, which is a type III spread whose form is shown in Fig. 7. Lines 46 and 48 do not belong to any λ-images of s_0 and s_2, so they must intersect ab at spreads which are λ-images of s_1, namely s_1, s_4, and s_7. Certainly 46 and 48 do not belong to $s_4 = s_1\lambda$, however.

Assume, for the moment, that $48 \in s_1$. Then $46 \notin s_1$ or s_4, so $46 \in s_7$; hence $(46)\,\lambda = 70 \in s_1$. Fig. 7 shows that then $36 \in s_1$, so $\{48, 70, 36\} \subset s_1$. This contradicts a consequence of (24), that $\{(56)\,\theta^2, (47)\,\theta^2\} = \{48, 36\} \subset s_2\theta^2$. Hence $48 \notin s_1$; instead, $48 \in s_7$, so $46 \in s_1$ and either 72, 75, or $78 \in s_1$. Since $78 \in s_0$ and $75 = (42)\lambda \in s_0\lambda$, $72 \in s_1$. Therefore,

$$s_1 = \{ab, c1, 46, 03, 72, 58\}. \tag{25}$$

From (25), $\{64, 58\} \subset s_1$, which contradicts a consequence of (24), that $\{(83)\theta, (47)\theta\} = \{64, 58\} \subset s_2\theta$. This completes the proof of the following theorem.

THEOREM 8.3. *No projective plane of order* 10 *has a collineation group of order* 9 *which fixes a* 12-*arc as a set.*

REFERENCES

1. E. F. ASSMUS, JR., The projective plane of order 10, *Math. Forsch. Inst. Oberwolfach*, Tagungsbericht 10 (1970).
2. E. F. ASSMUS, JR. AND H. F. MATTSON, JR., "Algebraic Theory of Codes, II," Final Report, AFCRL-71-0013, Air Force Cambridge Research Laboratories, Bedford, Mass., 1970.
3. E. F. ASSMUS, JR., H. F. MATTSON, JR., AND M. GUZA, Self-orthogonal Steiner systems and projective planes, *Math. Z.* **138** (1974), 89–96.
4. R. H. BRUCK, Construction problems in finite projective spaces, *in* "Finite Geometric Structures and Their Applications" (A. Barlotti, Ed.), Centro Internazionale Matematico Estivo, Bressanone, 1972.
5. R. H. BRUCK, personal communication.
6. R. BRUCK AND H. RYSER, The non-existence of certain finite projective planes, *Canad. J. Math.* **1** (1949), 88–93.
7. A. BRUEN AND J. C. FISHER, Blocking sets, *k*-arcs and nets of order ten, *Advances in Math.* **10** (1973), 317–320.
8. J. L. CARTER, "On the Existence of a Projective Plane of Order 10," Ph.D. thesis, University of California, Berkeley, 1975.
9. P. DEMBOWSKI, "Finite Geometries," Ergebnisse Series 44, Springer–Verlag, New York, 1968.
10. R. H. F. DENNISTON, Non-existence of a certain projective plane, *J. Austral. Math. Soc.* **10** (1969), 214–218.
11. M. M. GUZA, "Orthogonal Designs and Projective Planes of Order Congruent to Two Modulo Four," Ph.D. thesis, Lehigh University, 1973.
12. D. R. HUGHES, Collineations and generalized incidence matrices, *Trans. Amer. Math. Soc.* **86** (1957), 284–296.
13. D. R. HUGHES, Generalized incidence matrices over group algebras, *Illinois J. Math.* **1** (1957), 545–551.
14. D. HUGHES AND F. PIPER, "Projective Planes," Graduate Texts in Mathematics, Springer–Verlag, New York, 1973.

15. F. J. MacWilliams, N. J. A. Sloane, and J. G. Thompson, On the existence of a projective plane of order 10, *J. Combinatorial Theory Ser. A.* **14** (1973), 66–78.
16. S. H. Whitesides, Projective planes of order 10 have no collineations of order 11, *in* "Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing," pp. 515–526, February 1976, Baton Rouge, Utilitas Mathematica, Winnipeg.
17. S. H. Whitesides, "Collineations of Projective Planes of Order 10," Ph.D. thesis, University of Wisconsin, Madison, August 1975.